Math 055 Exam 1
Spring 2026

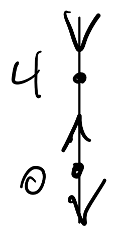
For full credit: Please show work using techniques from this course and use correct mathematical notation.

1. Consider the autonomous differential equation $\frac{dy}{dt} = y(4 - y)$

a. (2 pts) Find all equilibrium solutions (a.k.a critical numbers).

$y = 0$, $y = 4$

b. (2 pts) Sketch the phase portrait for this DE.



c. (2 pts) Determine their stability (asymptotically stable, unstable, semi-stable).

$y = 4$ is asymptotically stable
$y = 0$ is unstable

d. (1 pt) Describe the long-term behavior of solutions if:

$y(0) = 1$ $\qquad\qquad$ $y(0) = 5$

y tends upward toward 4

y tends downward toward 4

2. (7 pts) Solve $\frac{dy}{dx} = \frac{x}{1+y}$ subject to $y(0) = 3$.

$$\int (1+y)\,dy = \int x\,dx$$

$$y + \frac{1}{2}y^2 = \frac{1}{2}x^2 + C$$

$$y(0) = 3 \Rightarrow 3 + \frac{9}{2} = C \Rightarrow C = \frac{15}{2}$$

$$\boxed{\frac{1}{2}y^2 + y = \frac{1}{2}x^2 + \frac{15}{2}}$$

3. (9 pts) Solve $\frac{dy}{dx} + 2y = e^{-x}$ subject to $y(0) = 1$.

Linear. $\mu = e^{2x}$

$$\frac{d}{dx}(e^{2x}y) = e^{x} \Rightarrow e^{2x}y = e^{x} + c$$

$$y = e^{-x} + ce^{-2x}$$

$$y(0) = 1 \Rightarrow 1 = 1 + c \Rightarrow c = 0$$

$$\boxed{y = e^{-x}}$$

4. (10 pts) Solve $(2xy + 3)dx + (x^2 + 4y)dy = 0$. Then verify that your solution satisfies the differential equation.

$$M = 2xy + 3 \Rightarrow M_y = 2x$$

$$N = x^2 + 4y \Rightarrow N_x = 2x$$

The equation is exact.

$$f(x, y) = \int M dx = \int (2xy + 3) dx = x^2 y + 3x + h(y)$$

$$N = f_y = x^2 + h'(y) = x^2 + 4y$$

$$h'(y) = 4y \Rightarrow h(y) = 2y^2 + k$$

$$f(x, y) = C \Rightarrow \boxed{x^2 y + 3x + 2y^2 = C}$$

Check $f_x = 2xy + 3$, $f_y = x^2 + 4y$

so $f_x dx + f_y dy = 0$

$$\Rightarrow (2xy + 3)dx + (x^2 + 4y)dy = 0 \checkmark$$

5. (9 pts) Solve $\frac{dy}{dx} = \frac{(xy+y^2)}{x^2}$. (*Hint: Use a substitution.*)

Homogeneous:

$$x^2 dy - (xy + y^2)dx = 0$$

$$y = ux \Rightarrow dy = u\,dx + x\,du$$

$$\cancel{x^2}(u\,dx + x\,du) - (\cancel{x^2}u + \cancel{x^2}u^2)dx = 0$$

$$-u^2 dx + x\,du = 0$$

$$-u^{-2}du = -\frac{1}{x}dx$$

$$\frac{1}{u} = -\ln x + C$$

assuming $x > 0$

$$\frac{x}{y} = C - \ln x$$

$$\boxed{y = \frac{x}{C - \ln x}}$$

OR

Bernoulli:

$$\frac{dy}{dx} = \frac{1}{x}y + \frac{1}{x^2}y^2$$

$$\frac{dy}{dx} - \frac{1}{x}y = \frac{1}{x^2}y^2$$

$$u = y^{-1} \Rightarrow y = \frac{1}{u}$$

$$\frac{dy}{dx} = -\frac{1}{u^2}\frac{du}{dx}$$

$$-\frac{1}{u^2}\frac{du}{dx} - \frac{1}{x}\frac{1}{u} = \frac{1}{x^2}\frac{1}{u^2}$$

$$\frac{du}{dx} + \frac{1}{x}u = -\frac{1}{x^2}$$

$$\mu = e^{-\ln x} = x$$

$$\frac{d}{dx}(x\,u) = -\frac{1}{x}$$

$$xu = -\ln x + C$$

$$u = \frac{c - \ln x}{x}$$

$$\boxed{y = \frac{x}{c - \ln x}}$$

6. (10 pts) Solve $\frac{dy}{dx} + y = xy^2$.

Bernoulli w/ $n = 2$

$$u = y^{1-2} = y^{-1} \Rightarrow \frac{1}{u} = y$$

$$\frac{dy}{dx} = -\frac{1}{u^2}\frac{du}{dx}$$

$$-\frac{1}{u^2}\frac{du}{dx} + \frac{1}{u} = \frac{x}{u^2} \Rightarrow \frac{du}{dx} - u = -x$$

linear w/ $\mu = e^{-x}$

$$\int \frac{d}{dx}\left(e^{-x}u\right)dx = \int -xe^{-x}\,dx$$

$s = -x \quad dt = e^{-x}dx$

$ds = -dx \quad t = -e^{-x}$

$$e^{-x}u = xe^{-x} + e^{-x} + C$$

$$u = x + 1 + Ce^{x}$$

$$\Rightarrow \boxed{y = \left(x + 1 + Ce^{x}\right)^{-1}}$$

7. A tank initially contains 100 gallons of brine in which 50 lb of salt are dissolved. Brine containing 2 lb/gal flows in at 3 gal/min. The solution is well mixed and then drains at 2 gal/min.

a. (5 pts) Write a differential equation for the amount of salt $S(t)$. Be sure to include the initial condition.

Rate in $= 2\,\frac{lb}{gal}\left(3\,\frac{gal}{min}\right) = 6$ Rate out $= \frac{S\ lb}{(100+t)\,gal}\left(2\,\frac{gal}{min}\right)$

$$\boxed{\frac{dS}{dt} = 6 - \frac{2S}{100+t}, \quad S(0) = 50}$$

b. (4 pts) Solve for $S(t)$.

$$\frac{dS}{dt} + \frac{2}{100+t}S = 6 \quad \text{linear,} \quad \mu = e^{\int \frac{2}{100+t}dt} = e^{2\ln(100+t)} = (100+t)^2$$

$$\frac{d}{dt}\left[(100+t)^2 S\right] = 6(100+t)^2$$

$$(100+t)^2 S = 2(100+t)^3 + C$$

$$S = 2(100+t) + \frac{C}{(100+t)^2}$$

$$S(0) = 50 \Rightarrow 50 = 200 + \frac{C}{10000}$$

$$C = -150(10000)$$

$$\boxed{S(t) = 2(100+t) - \frac{1{,}500{,}000}{(100+t)^2}}$$

c. (3 pts) What happens to the volume and the salt concentration as $t \to \infty$?

As $t \to \infty$, volume $\to \infty$

Concentration is $\frac{lb}{gal} = \frac{S(t)}{100+t} = \frac{2(100+t) - \frac{1{,}500{,}000}{(100+t)^2}}{100+t}$

as $t \to \infty$, concentration $\to$ 2 lb/gal

8. A population satisfies $\frac{dP}{dt} = 0.5P\left(1 - \frac{P}{1000}\right)$. The logistic growth model that results from solving a related initial-value problem is $P(t) = \frac{1000}{1+4e^{-0.5t}}$. Use this information to answer the following.

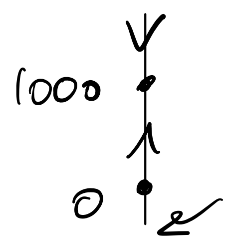
a. (1 pt) Identify the carrying capacity.

$P = 1000$

b. (2 pts) Find and classify all equilibrium solutions.

$P = 0$ and $P = 1000$
unstable　　　　Stable

c. (2 pts) Sketch the phase portrait for the DE.



not relevant for population

d. (3 pts) Determine the initial population.

$$P(0) = \frac{1000}{1+4} = 200$$

e. (2 pts) Determine the long-term behavior of the population.

As $t \to \infty$, $e^{-0.5t} \to 0$, so

$P \to 1000$ (the carrying capacity)